Estimation of Stress- Strength Reliability model using finite mixture of exponential distributions

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ABSTRACT:

In this paper considered a situation where stress and strength follow finite mixture of exponential distributions to find the reliability of a system. It has been studied when stress follow exponential distribution and strength follow finite mixture of exponential distributions and both stress-strength follow the finite mixture of exponential distributions. The general expression for the reliability of a system is obtained. The reliability is computed numerically for different values of the stress and strength parameters. We estimate the parameters of the reliability stress-strength models by the method of maximum likelihood estimation. The role of finite mixture of exponential distributions is illustrated using a real life data on time to death of two groups of leukaemia patients.

KEYWORDS: Exponential distribution, Finite mixture of exponential distributions, Maximum Likelihood Estimation, Reliability, Stress-Strength model.

1. INTRODUCTION

Reliability of a system is the probability that a system will adequately perform its intended purpose for a given period of time under stated environmental conditions [1]. In some cases system failures occur due to certain type of stresses acting on them. Thus system composed of random strengths will have its strength as random variable and the stress applied on it will also be a random variable. A system fails whenever an applied stress exceeds strength of the system. In a finite mixture model, the distribution of random quantity of interest is modelled as a mixture of a finite number of component distributions in varying proportions [2]. The flexibility and high degree of accuracy of finite mixture models have been the main reason for their successful applications in a wide range of fields in the biological physical and social sciences. The estimation of reliability based on finite mixture of pareto and beta distributions was studied by Maya, T. Nair (2007)[3].

In reliability theory, the mixture distributions are used for the analysis of the failure times of a sample of items of coherent laser used in telecommunication network. In an experiment, one hundred and three laser devices were operated at a temperature of 70 degree Celsius until all had failed. The experiment was run longer than one year before all the devices had failed, because most of the devices were extremely reliable. The sample thus consists of two distinct populations, one with a very short mean life and one with a much longer mean life. This can be considered as an example of a mixture of two exponential distributions with probability density function of the form

$$f(x) = p\lambda_1 \exp(-\lambda_1 x) + (1-p)\lambda_2 \exp(-\lambda_2 x), \quad 0 \le p \le 1, \lambda_i > 0, i = 1,2$$

The above model will be useful to predict how long all manufactured lasers should be life tested to assure that the final product contained no device from the infant mortality population.

In the present paper we discuss the statistical analysis of finite mixture of exponential distributions in the context of reliability theory. We give the definition and properties of the finite mixture of exponential distributions. We derive the reliability, when the strength X follows finite mixture of exponential and the stress Y takes exponential and finite mixture of exponential. We discuss estimation procedure for finite mixture of exponential distributions by the method of maximum likelihood estimation and also estimation of stress-strength reliability. We illustrate the method for a real data on survival times of leukaemia patients and finally give the conclusion.

II. STATISTICAL MODEL:

The assumptions taken in this model are

(i) The random variables X and Y are independent.

(ii) The values of stress and strength are non-negative.

If X denotes the strength of the component and Y is the stress imposed on it, then the reliability of the component is given by [1],

$$R = P(X > Y) = \int_0^\infty \left\{ \int_0^x g(y)dy \right\} f(x)dx \qquad (1)$$

where f(x) and g(y) are probability density functions of strength and stress respectively.

A finite mixture of exponential distribution with k-components can be represented in the form

$$f(x) = p_1\ f_1(x) + p_2\ f_2(x) + ... + p_k\ f_k(x) \qquad (2)$$

$$\text{where } p_i > 0, i = 1, 2, ..., k \quad \sum_{i=1}^{k} p_i = 1$$

The r^{th} moment of the mixture of two exponential distributions

$$E(x^r) = \int_0^\infty x^r \left[p_1 \lambda_1 \exp(-\lambda_1 x) + (1 - p_1) \lambda_2 \exp(-\lambda_2 x) \right] dx$$

$$= p_1\ \lambda_1 \frac{\overline{|(r+1)}}{\lambda_1^{r+1}} + (1-p_1)\lambda_2 \frac{\overline{|(r+1)}}{\lambda_2^{r+1}}$$

When r = 1, $E(x) = \dfrac{p_1}{\lambda_1} + \dfrac{1-p_1}{\lambda_2} \quad 0 < p_1 < 1, \lambda_i > 0, i = 1, 2$

When r = 2, $E(x^2) = \dfrac{2p_1}{\lambda_1^2} + \dfrac{2(1-p_1)}{\lambda_2^2}$

Thus the variance is given by $V(x) = \dfrac{p_1(2-p_1)}{\lambda_1^2} + \dfrac{(1-p_1)(2-p_1)}{\lambda_2^2} - \dfrac{2p_1(1-p_1)}{\lambda_1\ \lambda_2}$

In this paper we are considering two cases. They are

(1) Stress follows exponential distribution and strength follows finite mixture of exponential distributions.

(2) Stress and strength follows finite mixture of exponential distributions.

III. RELIABILITY COMPUTATIONS:

Let X be the strength of the k-components with probability density functions $f_i(x)$; i=1,2,..,k. Strength X follows finite mixture of exponential distribution with pdf

$$f_i(x) = p_i \lambda_i \exp(-\lambda_i x), \quad \lambda_i > 0, x > 0, p_i > 0, i = 1, 2, ..., k; \sum_{i=1}^{k} p_i = 1 \qquad (3)$$

3.1.Case(i) The stress Y follows exponential distribution:

When the stress Y follows exponential with pdf

$$g(y) = \lambda \exp(-\lambda y), y > 0, \lambda > 0$$

As a special case of (3) with k = 2, we have

$$f(x) = p_1 \lambda_1 \exp(-\lambda_1 x) + (1-p_1)\lambda_2 \exp(-\lambda_2 x), \lambda_i > 0, x > 0, (i = 1, 2)$$

And if X and Y are independent, then the reliability R from (1)

$$R_2 = \int_0^\infty \int_0^x (\lambda \exp(-\lambda y)) \left[p_1 \lambda_1 \exp(-\lambda_1 x) + (1-p_1)\lambda_2 \exp(-\lambda_2 x) \right] dydx$$

$$R_2 = 1 - p_1\left(\frac{\lambda_1}{\lambda+\lambda_1}\right) - (1-p_1)\left(\frac{\lambda_2}{\lambda+\lambda_2}\right) \qquad (4)$$

As a special case of (3) with k = 3, we have

$$f(x) = p_1\lambda_1\exp(-\lambda_1 x) + p_2\lambda_2\exp(-\lambda_2 x) + p_3\lambda_3\exp(-\lambda_3 x), \lambda_i > 0, x > 0, (i = 1,2,3); \sum_{i=1}^{3} p_i = 1$$

$$R_3 = \int_0^\infty\int_0^x (\lambda\exp(-\lambda y))\left[p_1\lambda_1\exp(-\lambda_1 x) + p_2\lambda_2\exp(-\lambda_2 x) + p_3\lambda_3\exp(-\lambda_3 x)\right]dydx$$

$$R_3 = 1 - p_1\left(\frac{\lambda_1}{\lambda+\lambda_1}\right) - p_2\left(\frac{\lambda_2}{\lambda+\lambda_2}\right) - p_3\left(\frac{\lambda_3}{\lambda+\lambda_3}\right) \qquad (5)$$

In general from (2), $f(x) = p_1\ f_1(x) + p_2\ f_2(x) + \ldots\ldots\ldots\ldots\ldots\ldots + p_k\ f_k(x)$

where $p_i > 0, i = 1,2,\ldots,k \quad \sum_{i=1}^{k} p_i = 1$

we get $$R_k = 1 - \sum_{i=1}^{k} p_i\frac{\lambda_i}{\lambda+\lambda_i} \qquad (6)$$

From table 1 and table 2 and figs.1 and 2, it is observed that if stress parameter increases then the value of reliability increases, if strength parameter increases then the value of reliability decreases.

3.2.Case (ii) The stress Y follows finite mixture of exponential distributions:

For k = 2, we have

$$f(x) = p_1\lambda_1\exp(-\lambda_1 x) + (1-p_1)\lambda_2\exp(-\lambda_2 x), \lambda_1, \lambda_2 > 0$$
$$g(y) = p_3\lambda_3\exp(-\lambda_3 x) + (1-p_3)\lambda_4\exp(-\lambda_4 x), \lambda_3, \lambda_4 > 0$$

And if X and Y are independent, then the reliability R from (1)

$$R_2 = \int_0^\infty\int_0^x \left[p_3\lambda_3\exp(-\lambda_3 y) + (1-p_3)\lambda_4\exp(-\lambda_4 y)\right]\left[p_1\lambda_1\exp(-\lambda_1 x) + (1-p_1)\lambda_2\exp(-\lambda_2 x)\right]dydx$$

$$= 1 - p_3\,p_1\frac{\lambda_1}{\lambda_1+\lambda_3} - p_3(1-p_3)\frac{\lambda_2}{\lambda_2+\lambda_3} - (1-p_3)\,p_1\frac{\lambda_1}{\lambda_1+\lambda_4} - (1-p_3)(1-p_1)\frac{\lambda_2}{\lambda_2+\lambda_4} \qquad (7)$$

For k = 3, we have

$$f(x) = p_1\lambda_1\exp(-\lambda_1 x) + p_2\lambda_2\exp(-\lambda_2 x) + p_3\lambda_3\exp(-\lambda_3 x), \lambda_i > 0, x > 0, (i = 1,2,3); \sum_{i=1}^{3} p_i = 1$$

Then

$$R_3 = \int_0^\infty\int_0^x \left[p_4\lambda_4\exp(-\lambda_4 y) + p_5\lambda_5\exp(-\lambda_5 y) + p_6\lambda_6\exp(-\lambda_6 y)\right]$$
$$\left[p_1\lambda_1\exp(-\lambda_1 x) + p_2\lambda_2\exp(-\lambda_2 x) + p_3\lambda_3\exp(-\lambda_3 x)\right]dydx$$

$$R_3 = 1 - p_4\,p_1\frac{\lambda_1}{\lambda_1+\lambda_4} - p_4\,p_2\frac{\lambda_2}{\lambda_2+\lambda_4} - p_4\,p_3\frac{\lambda_3}{\lambda_3+\lambda_4}$$

$$-p_5 p_1 \frac{\lambda_1}{\lambda_1+\lambda_5} - p_5 p_2 \frac{\lambda_2}{\lambda_2+\lambda_5} - p_5 p_3 \frac{\lambda_3}{\lambda_3+\lambda_5} - p_6 p_1 \frac{\lambda_1}{\lambda_1+\lambda_6} - p_6 p_2 \frac{\lambda_2}{\lambda_2+\lambda_6} - p_6 p_3 \frac{\lambda_3}{\lambda_3+\lambda_5}$$

$$R_3 = 1 - \sum_{j=i+3}^{6} \sum_{i=1}^{3} p_j \; p_i \frac{\lambda_i}{\lambda_i+\lambda_j} \tag{8}$$

In general from (2), $f(x) = p_1 \; f_1(x) + p_2 \; f_2(x) + ... + p_k \; f_k(x)$

where $p_i > 0, i = 1,2,...,k \quad \sum_{i=1}^{k} p_i = 1$

Then

$$R_k = 1 - \sum_{j=i+k}^{2k} \sum_{i=1}^{k} p_j \; p_i \frac{\lambda_i}{\lambda_i+\lambda_j} \tag{9}$$

From table 3 and table 4 and figs. 3 and 4, it is observed that if stress parameter increases then the value of reliability decreases, if strength parameter increases then the value of reliability increases.

3.3. Hazard Rate:

Let t denotes life time of a component with survival function S(t). Then the survival function of the model is obtained as

$$S(t) = p_1\lambda_1 \exp(-\lambda_1 t) + (1-p_1)\lambda_2 \exp(-\lambda_2 t), \quad \lambda_i > 0, \quad t > 0, \quad 0 < p_1 < 1$$

For the model the hazard rate h(t) is given by

$$h(t) = \frac{f(t)}{s(t)}$$

$$= \frac{p_1\lambda_1 \exp(-\lambda_1 t) + (1-p_1)\lambda_2 \exp(-\lambda_2 t)}{p_1 \exp(-\lambda_1 t) + (1-p_1)\exp(-\lambda_2 t)} \tag{10}$$

In general,

$$h(t) = \frac{\sum_{i=1}^{k} p_i \lambda_i \exp(-\lambda_i t)}{\sum_{i=1}^{k} p_i \exp(-\lambda_i t)} \tag{11}$$

Finite mixture of exponential possesses decreasing hazard rate and constant hazard rate depending upon the values of the parameters. Fig 5, show the behaviour of hazard rate at various time points.

IV. ESTIMATION OF PARAMETERS:

We estimate the parameters of the models by the method of maximum likelihood estimation.Consider the situation when there are only two sub populations with mixing proportions p_1 &($1-p_1$) and $f_1(x)$ and $f_2(x)$ are exponential densities with parameters $\lambda_1 \& \lambda_2$ respectively.

The likelihood function is given by

$$L(\lambda_1, \lambda_2, p_1 / \hat{y}) = \prod_{j=1}^{n} \left[p_1\lambda_1 \exp(-\lambda_1 y_j) + (1-p_1)\lambda_2 \exp(-\lambda_2 y_j) \right] \tag{12}$$

Where y_{ij} denoted the failure time of the j^{th} unit belonging to the i^{th} sub population $j=1,2,...n_i$; $i=1,2$ and

$$\hat{y} = \left\{ y_{11}, y_{12}, ...y_{1n_i}; y_{21}, y_{22}, ...y_{2n_i} \right\}$$

Maximization of log likelihood function of (12) w.r.t the parameters yields the following equation

$$L = \frac{\lfloor n}{\lfloor n_1 \lfloor n_2} p_1^{n_1} (1-p_1)^{n_2} \lambda_1^{n_1} \; \lambda_2^{n_2} \prod_{j=1}^{n_1} \exp(-\lambda_1 y_{1j}) \prod_{j=1}^{n_1} \exp(-\lambda_2 y_{2j})$$

$$L = c(\lambda_1 \, p_1)^{n_1} \left[(1-p_1)\lambda_2\right]^{n_2} \exp\left[-\sum_{j=1}^{n_1} y_{1j}\lambda_1 - \sum_{j=1}^{n_2} y_{2j}\lambda_2\right]$$

$$\log(L) = \log c + n_1 \log(p_1\lambda_1) + n_2 \log(1-p_1)\lambda_2 - \lambda_1 \sum_{j=1}^{n_1} y_{1j} - \lambda_2 \sum_{j=1}^{n_2} y_{2j}$$

$$\frac{n_1}{\lambda_1} - \sum_{j=1}^{n_1} y_{1j} = 0, \quad \frac{n_2}{\lambda_2} - \sum_{j=1}^{n_2} y_{2j} = 0, \quad \frac{n_1}{p_1} - \frac{n_2}{(1-p_1)} = 0$$

$$\Rightarrow \hat{\lambda}_1 = \frac{n_1}{\sum_{j=1}^{n_1} y_{1j}} \tag{13}$$

$$\Rightarrow \hat{\lambda}_2 = \frac{n_2}{\sum_{j=1}^{n_2} y_{2j}} \tag{14}$$

$$\Rightarrow \hat{p}_1 = \frac{n_1}{n_1 + n_2} = \frac{n_1}{n} \quad where \quad n = n_1 + n_2 \tag{15}$$

The above results can be generalized for any k, giving the following estimators

$$\hat{p}_1 = \frac{n_1}{n} \quad \& \quad n = \sum_{i=1}^{k} n_1 \tag{16}$$

$$\Rightarrow \hat{\lambda}_i = \frac{n_i}{\sum_{j=1}^{n_i} y_{ij}} \tag{17}$$

4.1. Estimation of Stress – Strength reliability:

(i) When the strength X follows finite mixture of exponential distributions with parameters λ_i and p_i and the stress Y follows exponential distribution with parameter λ, then the M.L.E of R is given as

$$\hat{R}_k = 1 - \sum_{i=1}^{k} \hat{p}_i \frac{\hat{\lambda}_i}{\lambda + \hat{\lambda}_i} \tag{18}$$

(ii) When the strength X follows finite mixture of exponential distributions with parameters λ_i and p_i and the stress Y follows finite mixture of exponential distribution with parameter λ_j and p_j then the M.L.E of R is given as

$$\hat{R}_k = 1 - \sum_{j=i+k}^{2k} \sum_{i=1}^{k} \hat{p}_j \, \hat{p}_i \frac{\hat{\lambda}_i}{\hat{\lambda}_i + \hat{\lambda}_i} \tag{19}$$

V. DATA ANALYSIS:

We consider a data on time to death of two groups of leukaemia patients which is given in Table 5 (see Feigl and Zelen, 1965) to illustrate the procedure of estimation. We then estimate the parameters using M.L.E technique. Table 6 provides the values of the estimates by M.L.E method. Table 7 provides the maximum likelihood estimate of survival function at various time points. Table 8 provides the hazard rate function at various time points.

Case (i) Stress has exponential distribution and Strength has mixture two of exponential distributions:

Table 1

$P_1 = \lambda_1 = \lambda_2$	λ	R
0.1	0.1	0.5
0.1	0.2	0.666667
0.1	0.3	0.75
0.1	0.4	0.8
0.1	0.5	0.833333
0.1	0.6	0.857143
0.1	0.7	0.875
0.1	0.8	0.888889
0.1	0.9	0.9
0.1	1	0.909091

Table 2

P_1	λ	$\lambda_1 = \lambda_2$	R
0.1	0.7	0.1	0.875
0.1	0.7	0.2	0.777778
0.1	0.7	0.3	0.7
0.1	0.7	0.4	0.636364
0.1	0.7	0.5	0.583333
0.1	0.7	0.6	0.538462
0.1	0.7	0.7	0.5
0.1	0.7	0.8	0.466667
0.1	0.7	0.9	0.4375
0.1	0.7	1	0.411765

Case(ii) Stress-Strength has mixture two of exponential distributions:

Table 3

$P_1 = P_3$	$\lambda_1 = \lambda_2$	$\lambda_3 = \lambda_4$	R
0.1	0.1	0.7	0.875
0.1	0.2	0.7	0.777778
0.1	0.3	0.7	0.7
0.1	0.4	0.7	0.636364
0.1	0.5	0.7	0.583333
0.1	0.6	0.7	0.538462
0.1	0.7	0.7	0.5
0.1	0.8	0.7	0.466667
0.1	0.9	0.7	0.4375
0.1	1	0.7	0.411765

Table 4

$P_1 = P_3 = \lambda_1 = \lambda_2$	$\lambda_3 = \lambda_4$	R
0.1	0.1	0.5
0.1	0.2	0.666667
0.1	0.3	0.75
0.1	0.4	0.8
0.1	0.5	0.833333
0.1	0.6	0.857143
0.1	0.7	0.875
0.1	0.8	0.888889
0.1	0.9	0.9
0.1	1	0.909091

Table 5

Survival times of leukaemia patients

AG +ve	143	56	26	134	16	65	156	100	39	1	5	65	22	1	4	108	121
AG –ve	2	3	8	7	16	22	3	4	56	65	17	4	3	30	4	43	

Table 6

Estimates of parameters of survival times of leukaemia patients.

$\hat{\lambda}_1$	0.016
$\hat{\lambda}_2$	0.055
$\hat{p}_1$	0.515

Table 7

Maximum likelihood estimate of survival probability at various time points

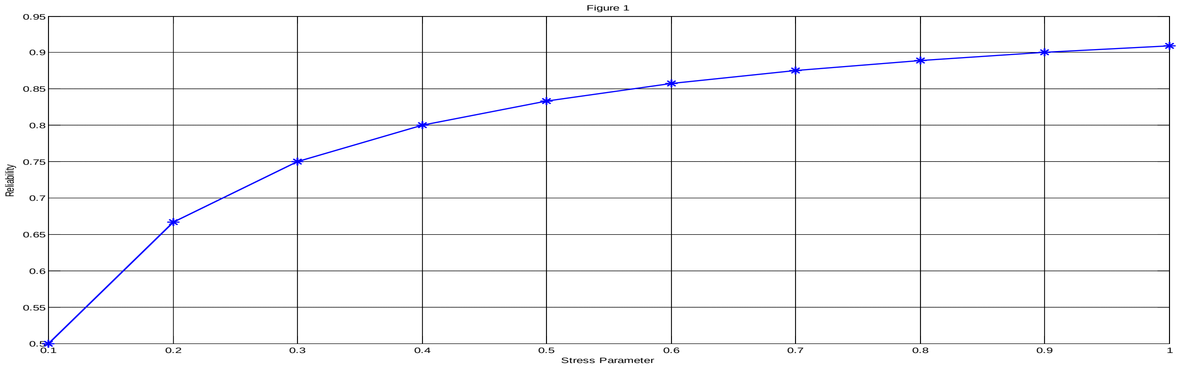
T	1	10	50	75	100	120
$\hat{S}(t)$	0.9659	0.7187	0.2624	0.1630	0.1060	0.0762

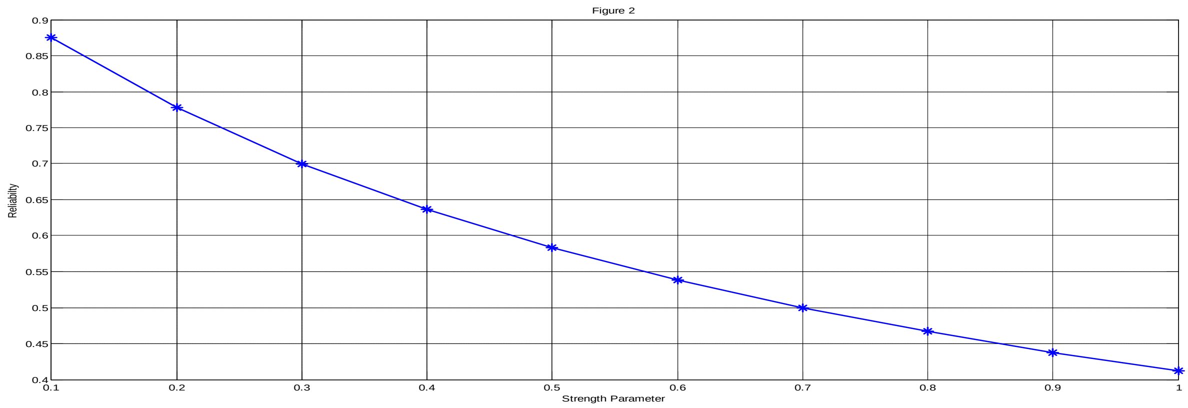
Table 8

Hazard rate at various time points

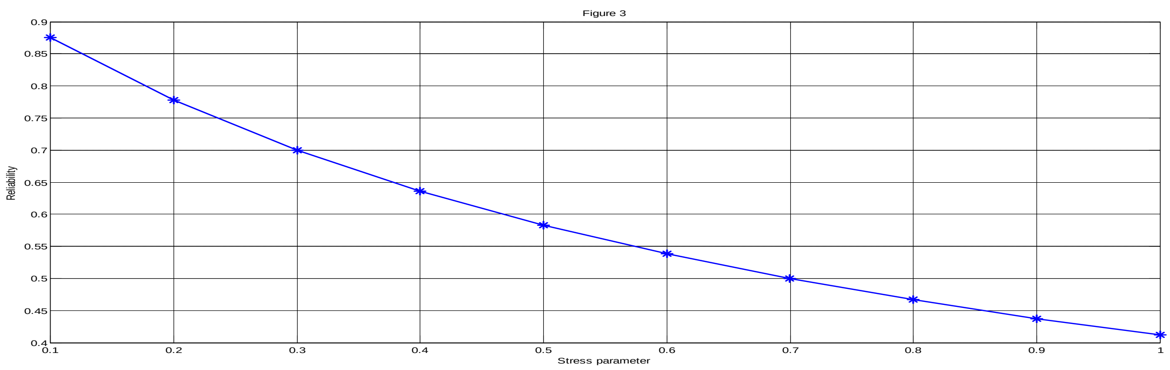
T	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
h(t)	0.5867	0.5501	0.5158	0.4836	0.4535	0.4252	0.3988	0.3740	0.3507	0.3289

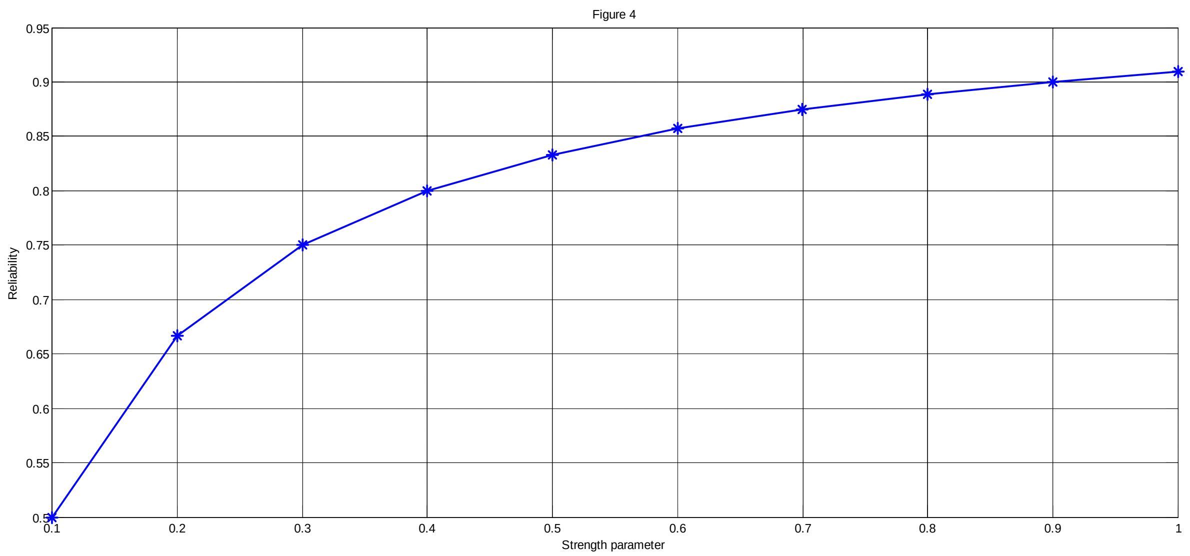
Case (i) Stress has exponential distribution and Strength has mixture two of exponential distributions:



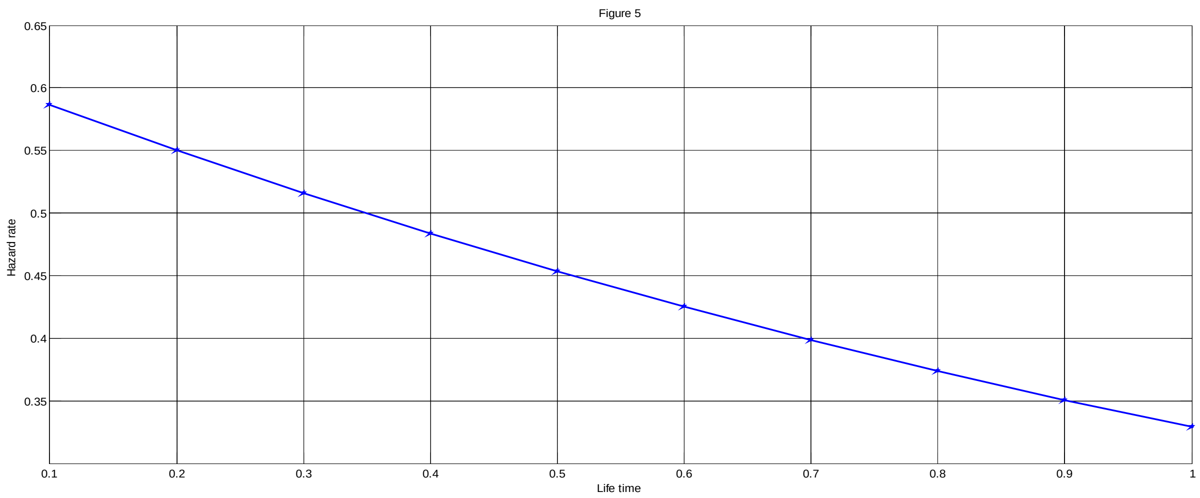


Case(ii) Stress-Strength has mixture two of exponential distributions:





Hazard rate function:



VI. CONCLUSION:

The role of finite mixture of exponential distributions in reliability analysis is studied. We derive the reliability, when the strength X follows finite mixture of exponential and the stress Y takes exponential and finite mixture of exponential. It has been observed by the computations and graphs, in case(i) if stress parameter increases then the value of reliability increases, if strength parameter increases then the value of reliability decreases. Where as in case(ii) if stress parameter increases then the value of reliability decreases, if strength parameter increases then the value of reliability increases. We developed estimates of parameters using Maximum likelihood estimation. The role of finite mixture of exponential distributions is illustrated using a real life data on time to death of two groups of leukaemia patients.

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